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Exact static solutions of a generalized discrete ϕ^4 model including short-periodic solutions

Avinash Khare[1], **Sergey V Dmitriev**[2] **and Avadh Saxena**[3]

[1] Institute of Physics, Bhubaneswar, Orissa 751005, India
[2] Institute for Metals Superplasticity Problems RAS, 450001 Ufa, Khalturina 39, Russia
[3] Center for Nonlinear Studies and Theoretical Division,
Los Alamos National Laboratory, Los Alamos, NM 87545, USA

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Abstract

We carry out a comprehensive analysis of a generalized discrete ϕ^4 model, of which virtually all ϕ^4 models discussed in the literature are particular cases. For this model we construct the exact solutions in the form of the basic Jacobi elliptic, hyperbolic and sine functions, and also give a list of short-periodic and even aperiodic solutions. Some of those solutions coincide with the known ones, others generalize the existing solutions and the rest of them are new. We then discuss the relation between the models supporting exact static solutions and the two-point maps. In particular, we show that some of the short-periodic and sine solutions can be found from factorized difference equations and even from a set of two difference equations, one of the first and another of the second order. Particular attention is paid to the discussion of the exceptional discrete (ED) models defined as models supporting the translationally invariant (TI) static solutions that can be placed arbitrarily with respect to the lattice. We show that some of the derived short-periodic solutions are TI ones while the others are not. For the TI static solutions we demonstrate the existence of the translational Goldstone mode for any location of the solution with respect to the lattice. We then analyze numerically the stability and other properties of the TI kink solutions. In conclusion, we divide the ED models into two classes: the ED I models support a two-parameter set of TI static solutions, while the ED II models support only a one-parameter set of such solutions.

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1. Introduction and setup

Discrete nonlinear models appear in a number of important applications, for example, in the physics of plastic deformation [1], in optics for light pulses moving in optical waveguides or in photorefractive crystal lattices (see, e.g., [2]) and in atomic physics for Bose–Einstein

condensates moving through optical lattice potentials (see, e.g., [3] for a recent review), to name a few. Discrete nonlinear equations emerge not only as the models describing discrete physical systems but also in developing efficient numerical approaches to the analysis of the partial derivative nonlinear equations in various fields of physics (see, e.g., [4–6]).

Recently, several exceptional discretizations of nonlinear partial derivative equations have been developed such that they admit translationally invariant (TI) static solutions (i.e., static solutions with an arbitrary shift along the lattice). Such exceptional discrete (ED) models have been constructed and investigated for the Klein–Gordon field [7–18] and for the nonlinear Schrödinger equation [19–24]. In the Hamiltonian ED models [8, 12], TI solutions do not experience the Peierls–Nabarro (PN) potential [1]. In the non-Hamiltonian ED models, the height of the Peierls–Nabarro barrier is path dependent but there exists a continuous path along which the work required for a quasi-static shift of the solution along the lattice is zero [16]. Static solitary waves in such lattices possess the translational Goldstone mode [10, 11, 19–21]. Exact solutions moving with selected velocities have also been constructed [20, 21, 25, 26].

ED models are interesting from the mathematical point of view because they admit exact static or even moving solutions (the relation between the former ones and the latter ones has been established in our recent work [26]) and are closely related to the theory of integrable maps [27–29]. ED models are also interesting from the physical standpoint, because the TI static solutions in such discrete models are not trapped by the lattice and they can be accelerated by even a weak external field [11]. In contrast to that, the static coherent structures in non-ED lattices, being located at the bottom of the PN potential, can be accelerated by the field sufficiently strong to overcome the PN barrier.

In the present study, we discuss not only Jacobi elliptic function (JEF) solutions but also sine and other short-periodic and even aperiodic solutions. Exact, extended, sinusoidal solutions of the lattice equations have been recently found by several authors [30–34]. It has been proposed that such solutions can be used to construct approximate large-amplitude *localized* solutions by truncating the sine solutions [31, 35].

For some of the ED models it has been demonstrated that they conserve momentum [7] or energy (Hamiltonian) [8, 12] (see also [20, 23]). However, we do not know a TI model conserving both momentum and Hamiltonian and, for the Klein–Gordon lattices with classically defined momentum, it was proved that these two conservation laws are mutually exclusive [14].

As far as ED models are concerned, it is believed that they should possess the following properties: (i) they admit static solutions which can be placed anywhere with respect to the lattice; this can be associated with the absence of the PN barrier [16]. (ii) Static version of ED models are integrable, i.e., the three-point static problems are reducible to two-point static problems which can be viewed as a nonlinear map from which static solutions can be constructed iteratively (in this study we will show that non-integrable three-point static problems can also have particular TI solutions derivable from factorized equations). (iii) Static solutions in ED models possess the translational Goldstone mode with zero frequency for any x_0 (see section 6).

A prototype class of discrete models, relevant to a variety of applications are the so-called discrete ϕ^4 models which feature a cubic nonlinearity. The purpose of this paper is to study in detail several issues related to ED models. In particular, we consider a rather general discrete ϕ^4 model with cubic nonlinearity which is invariant under the interchange of ϕ_{n+1} and ϕ_{n-1}

$$\ddot{\phi}_n = \frac{1}{h^2}(\phi_{n+1} + \phi_{n-1} - 2\phi_n) + \lambda\phi_n - A_1\phi_n^3 - \frac{A_2}{2}\phi_n^2(\phi_{n+1} + \phi_{n-1}) - \frac{A_3}{2}\phi_n\left(\phi_{n+1}^2 + \phi_{n-1}^2\right) - A_4\phi_n\phi_{n+1}\phi_{n-1} - \frac{A_5}{2}\phi_{n+1}\phi_{n-1}(\phi_{n+1} + \phi_{n-1}) - \frac{A_6}{2}\left(\phi_{n+1}^3 + \phi_{n-1}^3\right), \tag{1}$$

with the model parameters satisfying the constraint

$$\sum_{k=1}^{6} A_k = \lambda. \tag{2}$$

In equation (1), $\phi_n(t)$ is the unknown function defined on the lattice $x_n = hn$ with the lattice spacing $h > 0$ and overdot means derivative with respect to time t. Without any loss of generality it is sufficient to consider $\lambda = 1$ or $\lambda = -1$.

If model parameters A_k are constant (i.e., independent of h), condition of equation (2) ensures that, in the continuum limit, equation (1) reduces to the ϕ^4 equation

$$\phi_{tt} = \phi_{xx} + \lambda\phi(1 - \phi^2). \tag{3}$$

On the other hand, if model parameters A_k are functions of h, then the continuum limit can be different from equation (3) even when equation (2) holds.

Static form of equation (3) has the first integral

$$\phi_x^2 - \frac{\lambda}{2}(1 - \phi^2)^2 + C = 0, \tag{4}$$

with the integration constant C.

So far as we are aware of, all the discrete ϕ^4 models discussed in the literature, under static consideration, are special cases of the general model, equation (1). Some of these models are:

Model 1. Only A_1 nonzero with other A_k being equal to zero results in the classical discretization of equation (3) that has received a great deal of attention from the researchers in various fields. This model is not an ED one and it will not be discussed further here.

Model 2. $A_1 = A_3/2 = A_4 = \lambda\delta$, $A_5 = A_6 = 2\lambda\gamma$, $A_2 = \lambda(1 - 4\gamma - 4\delta)$ with arbitrary γ and δ. This non-Hamiltonian ED model (for arbitrary γ and δ) conserves momentum [7]

$$P_1 = \sum_n \dot{\phi}_n(\phi_{n+1} - \phi_{n-1}). \tag{5}$$

Static version of this model (with the omitted inertia term $\ddot{\phi}_n$) has the first integral [10]

$$\begin{aligned} U(\phi_{n-1}, \phi_n) \equiv{}& (\phi_n - \phi_{n-1})^2 + \Lambda\phi_n\phi_{n-1} - \Lambda\gamma\left(\phi_n^4 + \phi_{n-1}^4\right) \\ & - \Lambda\delta\phi_n\phi_{n-1}\left(\phi_n^2 + \phi_{n-1}^2\right) + \Lambda\left(2\gamma + 2\delta - 1/2\right)\phi_n^2\phi_{n-1}^2 - C\Lambda/2 = 0, \end{aligned} \tag{6}$$

where C is the integration constant and Λ is defined (throughout the paper) by

$$\Lambda = \lambda h^2, \tag{7}$$

from which any static solution to equation (1) can be constructed iteratively, starting from any admissible value of ϕ_0 and solving at each step the algebraic problem. This is so because equation (1) is nothing but

$$\ddot{\phi}_n = \frac{U(\phi_n, \phi_{n+1}) - U(\phi_{n-1}, \phi_n)}{\phi_{n+1} - \phi_{n-1}}. \tag{8}$$

Equation (6) is the discretized first integral (DFI) [16], i.e., in the continuum limit ($h \to 0$) it reduces to equation (4). In this complete form, the model was first published independently in [10, 13], although in the latter work any relation to the DFI was not observed and only cubic nonlinearity was treated, while in the former work, ED models were constructed for a general polynomial nonlinearity. Note that Model 2 with $\gamma = \delta = 0$ is the Bender–Tovbis model [9]. In the framework of the DFI approach [16], almost the entire space of static solutions supported by this model was described and many of those solutions were expressed in terms of

JEF [16] (see also [36]). We also note that Model 2 with $\delta = 0$ and $\gamma = 1/4$ is the Kevrekidis model [7].

Model 3. Discrete ϕ^4 model

$$\ddot{\phi}_n = \frac{1}{h^2}(\phi_{n-1} - 2\phi_n + \phi_{n+1}) + \frac{\lambda\left(\phi_n - \phi_n^3\right)}{1 - \Lambda\phi_n^2/2}, \tag{9}$$

discovered in [12] does not belong to equation (1) but its static problem coincides with that of the Bender–Tovbis model [12, 16]. Some very special features of this ED model are the conservation of energy and the on-site discretization of the nonlinear term. In all other ED ϕ^4 models derived so far, the nonlinear term is discretized on the three neighboring nodes (i.e., lattice sites).

Model 4. With only A_4 nonzero and other A_k being equal to zero, one arrives at the non-Hamiltonian ED model derived by Barashenkov *et al* [13] and referred to as BOP. This model conserves the momentum defined by [18]

$$P_2 = \sum_n \dot{\phi}_n(\phi_{n+2} - \phi_{n-2}). \tag{10}$$

The first integral of the static version of this model has been found in [18], where an almost complete set of static solutions supported by this model were derived and many of those solutions were expressed in terms of JEF.

Model 5. Taking $A_1 = 2\lambda/9, A_2 = A_3 = \lambda/3, A_4 = A_5 = 0, A_6 = \lambda/9$ one gets the Hamiltonian of the Speight and Ward (SW) model [8]. This model supports TI static kinks derivable from the two-point map [8]

$$v(\phi_{n-1}, \phi_n) \equiv \frac{\phi_n - \phi_{n-1}}{H} - \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{2}}\left(\phi_{n-1}^2 + \phi_{n-1}\phi_n + \phi_n^2\right) = 0, \quad H^2 = \frac{6\Lambda}{6 - \Lambda}. \tag{11}$$

Note that this map is defined in case $0 < \Lambda < 6$. To get this map, one has to set $C = 0$ in equation (4) and present it as $\phi_x \pm \sqrt{\lambda/2}(1 - \phi^2) = 0$. Equation (11) is a discrete version of the last equation. It is not known if this model supports static TI solutions other than the kink. It is also not known if this model has the first integral of the static problem apart from the case of $C = 0$. In the present study, we will provide evidence that the answer to the second question is negative (see section 7.1) but we were able to find other TI solutions to this model, see equation (25).

Model 6. $A_1 = 4\lambda\alpha(\gamma + \beta), A_2 = 4\lambda[2\alpha^2 + \gamma^2 + \beta(\gamma - \alpha)], A_3 = 4\lambda\alpha(2\gamma + \beta), A_4 = 4\lambda\gamma(\alpha - \beta), A_5 = 4\lambda\alpha(\alpha - \beta), A_6 = 4\lambda\alpha^2$, with two free parameters α and β with $\gamma = 1/2 - 2\alpha$. This model was also proposed by Barashenkov *et al* [13]. This model includes as special cases the Bender–Tovbis model (at $\alpha = \beta = 0$), the Model 4 (at $\alpha = 0$ and $\beta = -1/2$) and also the Model 5 (at $\alpha = \beta = 1/6$).

Model 7. $A_1 = 0, A_2 = 2\lambda(1/2 - \beta), A_3 = \lambda\sigma(4 + h^2), A_4 = 2\lambda\beta + \lambda\sigma(4 + h^2), A_5 = -8\lambda\sigma, A_6 = 0$, with two free parameters β and σ. This model was also proposed in [13]. Note that the coefficients A_3 and A_4 in this model are h-dependent and that the constraint equation (2) for this model is satisfied only in the continuum limit (i.e. for $h = 0$).

In this paper, we shall discuss five more models with cubic nonlinearities.

Model 8. $A_1 = 4\alpha_1\lambda, A_2 = 6\alpha_2\lambda, A_3 = 4\alpha_3\lambda, A_6 = 2\alpha_2\lambda, A_4 = A_5 = 0$, with $\alpha_1 + 2\alpha_2 + \alpha_3 = 1/4$. This model has two free parameters α_i. In section 2, the Hamiltonian for this model will be given.

Model 9. Only A_3 and A_5 nonzero.

Model 10. Only A_2, A_3 and A_5 nonzero.

Model 11. Only A_3, A_4 and A_5 nonzero.

Model 12. Only A_2, A_3, A_4 and A_5 nonzero.

We note that the exact tanh solution to the Models 9–12 has been derived in [13]. It will be demonstrated that the Models 9–12 support the TI static solutions and thus they are ED models.

The paper is organized as follows. In section 2, we discuss the subclasses of model equation (1) that support different conservation laws. In section 3, we report on a number of TI static solutions to equation (1) expressed in closed analytical form. All seven cases, when model equation (1) supports the exact static JEF solutions, are described. Two of the seven cases have been previously studied in the literature, and for the remaining five cases, basic JEF solutions are given here together with their hyperbolic function limits. We also obtain a periodic sine solution for the general five-parameter model as given by equation (1). Section 4 presents a number of exact short-periodic static solutions. In section 5, we discuss the two-point maps for some of the ED models and derive a map for the model in case only A_2 and A_4 are nonzero. Goldstone translational modes of the TI static solutions are discussed in section 6. Numerical results that illustrate some important properties of the TI static solutions are presented in section 7. In addition to the discussion, conclusions and future challenges are described in section 8.

2. Momentum and energy conservation

As noted above, for the discrete model of equation (1), the momentum operator as given by equation (5) is conserved in Model 2 (for arbitrary values of γ and δ) and in some other models reducible to the form of Model 2. On the other hand, as was already mentioned, the momentum defined by equation (10) is conserved for equation (1) in Model 4, i.e. in case only A_4 is nonzero.

In the case of Model 8, equation (1) can be obtained from the Hamiltonian

$$H = \sum_n \left[\frac{\dot{\phi}_n^2}{2} + \frac{(\phi_n - \phi_{n-1})^2}{2h^2} + \frac{\lambda}{4} - \frac{\lambda}{2}\phi_n^2 + \alpha_1\phi_n^4 + \alpha_2\phi_n\phi_{n-1}\left(\phi_n^2 + \phi_{n-1}^2\right) + \alpha_3\phi_n^2\phi_{n-1}^2 \right], \tag{12}$$

and hence energy is conserved in this model. Note that the Hamiltonian model has two free parameters. The SW Model 5 is a special case of this model in case $\alpha_1 = \alpha_2 = 1/18$ and $\alpha_3 = 1/12$.

3. Translationally invariant JEF, hyperbolic and trigonometric static solutions

All solutions described in this section have either the form of

$$\phi_n = SAf[h\beta(n + x_0), m], \tag{13}$$

or the form of

$$\phi_n = SAg[h\beta(n + x_0)]. \tag{14}$$

In equation (13), f denotes one of the three basic JEF functions, sn, cn or dn and $0 \leqslant m \leqslant 1$ is the JEF modulus. In equation (14), g means one of the two hyperbolic functions tanh or sech or the sine function. The tanh solution is obtained from the sn solution in the limit of $m = 1$, while the sech solution is obtained in this limit equivalently from cn or dn solutions. Ideally, the trigonometric solutions should be the $m \to 0$ limit of the JEF solutions. However, the relevant JEF identities we have used [37] vanish in this limit and thus the trigonometric solutions must be derived separately.

In equations (13) and (14), x_0 is an arbitrary shift, and $S = 1$ for the non-staggered and $S = (-1)^n$ for the staggered solutions. Expressions that relate the solution parameters A and β to the model parameters A_k and, where applicable, the relations between A_k, are given in what follows.

We shall first discuss the JEF solutions as well as the hyperbolic solutions which follow from the JEF solutions and later on we shall discuss the sine solutions.

3.1. JEF and hyperbolic solutions

We shall now show that JEF solutions can be obtained for the discrete model of equation (1) in case $A_1 = A_6 = 0$ in the following seven cases: (i) only A_2 nonzero (i.e., Model 2 at $\gamma = \delta = 0$); (ii) only A_4 nonzero (i.e., Model 4); (iii) only A_2 and A_4 nonzero (i.e., Model 7 at $\sigma = 0$); (iv) only A_3 and A_5 nonzero (i.e., Model 9); (v) A_2, A_3 and A_5 nonzero (Model 10); (vi) A_3, A_4 and A_5 nonzero (Model 11); (vii) A_2, A_3, A_4 and A_5 nonzero (Model 12).

The JEF solutions have already been reported in case (i) in [16, 36] and in case (ii) in [18]. In this paper, we report on the JEF solutions for cases (iii) to (vii).

Note that if $\phi_n(t)$ is a solution to equation (1), then the staggered solution $(-1)^n\phi_n(t)$ is also a solution to the same equation, but with the coefficients A_1, A_3 and A_4 having the opposite signs, and further $2 - \Lambda$ is to be replaced by $\Lambda - 2$. To make the presentation of the results as compact as possible, in most cases, we shall therefore not give the parameters for the staggered solutions. Further, for the same reason, we give JEF solutions only for the general Model 12, the results for all other models can be easily obtained from here. However, for few models we do give conditions for the existence of the kink solution. In some cases, the conditions for the existence of the staggered hyperbolic function solutions are not trivially derived from those for the nonstaggered solutions, and these conditions can be found in the extended version of this work [38].

Recall the following definitions for the additional JEF functions through the three basic JEF functions,

$$\mathrm{ns}(x, m) = 1/\mathrm{sn}(x, m), \quad \mathrm{cs}(x, m) = \mathrm{cn}(x, m)/\mathrm{sn}(x, m), \quad \mathrm{ds}(x, m) = \mathrm{dn}(x, m)/\mathrm{sn}(x, m). \tag{15}$$

For the sake of brevity, in the following, we will drop the JEF modulus m in all JEF functions, e.g., instead of writing $\mathrm{sn}(h\beta, m)$ we will simply write $\mathrm{sn}(h\beta)$ and so on.

Finally, we introduce the notation

$$T = \tanh^2(h\beta). \tag{16}$$

Model 12: nonzero A_2, A_3, A_4 and A_5 with $A_1 = A_6 = 0$. The solution equation (13) with $f = \mathrm{sn}$ exists provided

$$A_5\,\mathrm{ns}(2h\beta) = \mp A_3\,\mathrm{ns}(h\beta),$$

$$\frac{2m}{A^2h^2} = \pm 2A_4\,\mathrm{ns}(h\beta)\,\mathrm{ns}(2h\beta) \mp A_3\,\mathrm{cs}(h\beta)\,\mathrm{ds}(h\beta) + A_2\,\mathrm{ns}^2(h\beta)$$
$$+ A_5[\mathrm{ns}^2(2h\beta) - \mathrm{cs}(2h\beta)\,\mathrm{ds}(2h\beta)],$$
$$\pm\frac{(2-\Lambda)}{A^2h^2} = \pm(A_4 - A_3)\,\mathrm{ns}^2(h\beta) + A_2\,\mathrm{cs}(h\beta)\,\mathrm{ds}(h\beta), \tag{17}$$

where (and in the following) the upper (lower) sign corresponds to the nonstaggered (staggered) solution. In the limit $m = 1$, the nonstaggered solution goes over to the kink solution equation (14) with $g = \tanh$, $S = 1$, satisfying

$$A^2 = 1, \qquad A_5T = -2A_3 - A_5, \qquad \Lambda = T\left[2 - h^2A_4 + \frac{h^2A_3(3-T)}{(1+T)}\right]. \tag{18}$$

Note that this kink solution exists for any value of Λ.

On the other hand, the solution equation (13) with $f = \mathrm{dn}$ exists in this model provided

$$A_5\,\mathrm{cs}(2h\beta) = \mp A_3\,\mathrm{cs}(h\beta),$$
$$\frac{2}{A^2h^2} = \mp 2A_4\,\mathrm{cs}(h\beta)\,\mathrm{cs}(2h\beta) \pm A_3\,\mathrm{ds}(h\beta)\,\mathrm{ns}(h\beta)$$
$$- A_2\,\mathrm{cs}^2(h\beta) + A_5[\mathrm{ns}(2h\beta)\,\mathrm{ds}(2h\beta) - \mathrm{cs}^2(2h\beta)],$$
$$\pm(2-\Lambda)/A^2h^2 = \pm(A_3 - A_4)\,\mathrm{cs}^2(h\beta) - A_2\,\mathrm{ns}(h\beta)\,\mathrm{ds}(h\beta). \tag{19}$$

Yet another exact solution is given by equation (13) with $f = \mathrm{cn}$ satisfying

$$A_5\,\mathrm{ds}(2h\beta) = \mp A_3\,\mathrm{ds}(h\beta),$$
$$\frac{2m}{A^2h^2} = \mp 2A_4\,\mathrm{ds}(h\beta)\,\mathrm{ds}(2h\beta) \pm A_3\,\mathrm{cs}(h\beta)\,\mathrm{ns}(h\beta)$$
$$- A_2\,\mathrm{ds}^2(h\beta) + A_5[\mathrm{ns}(2h\beta)\,\mathrm{cs}(2h\beta) - \mathrm{ds}^2(2h\beta)],$$
$$\pm\frac{(2-\Lambda)m}{A^2h^2} = \pm(A_3 - A_4)\,\mathrm{ds}^2(h\beta) - A_2\,\mathrm{ns}(h\beta)\,\mathrm{cs}(h\beta). \tag{20}$$

In the limit $m = 1$, both the solutions (19) and (20) go over to the pulse solution equation (14) with $g = \mathrm{sech}$, $S = 1$ satisfying

$$\Lambda = -2[\cosh(h\beta) - 1] < 0, \qquad h^2(A_2 + A_4) = (1 + |\Lambda|)h^2A_3 - |\Lambda|,$$
$$A_5 = -(|\Lambda| + 2)A_3, \qquad A^2 = \frac{(|\Lambda|+2)(|\Lambda|+4)}{2(2 - h^2A_2 - 2h^2A_3)}. \tag{21}$$

Note that the pulse solution exists only if $\Lambda < 0$ while staggered pulse solution exists only if $\Lambda > 4$.

Using equations (17)–(21) and the constraint equation (2), one can obtain the kink and the pulse solutions in Model 7 (at $\sigma = 0$) and in Models 9–11. Below we specify only the kink solution and that too only for the three particular cases considered later in section 7.2.

Model 7 with $\sigma = 0$: only A_2 and A_4 nonzero. In this model the nonstaggered kink solution, equation (14), with $g = \tanh$, $S = 1$ exists provided

$$A^2 = 1, \qquad h^2A_4T = 2T - \Lambda, \qquad A_2 = \lambda - A_4. \tag{22}$$

Note that this solution is valid for any value of Λ.

Model 9: only A_3 and A_5 nonzero. In this model, the kink solution equation (14) with $g = \tanh$, $S = 1$ exists provided

$$A^2 = 1, \qquad 2A_3 = -A_5(1+T), \qquad \Lambda(1 + 2T - T^2) = 2T(1-T). \tag{23}$$

The kink solution exists only if $0 < \Lambda < (2 - \sqrt{2})/2$.

Model 11: only A_3, A_4 and A_5 are nonzero. In this case, the kink solution equation (14) with $g = \tanh$, $S = 1$ exists provided the parameters satisfy

$$A^2 = 1, \qquad 2h^2A_3T = (1+T)[(\Lambda-2)T+\Lambda],$$
$$2h^2A_4T = \Lambda + 2(\Lambda-1)T - (\Lambda-2)T^2, \qquad h^2A_5T = -(\Lambda-2)T - \Lambda. \tag{24}$$

This solution is valid for any Λ.

3.2. Trigonometric solution

Unlike the JEF and the hyperbolic solutions, the static TI trigonometric solution of the form of equation (14) with $g = \sin$ exists even when all the six parameters A_i are nonzero, namely, it exists under the following two conditions:

$$\pm(\Lambda-2) + 2\cos(h\beta) = h^2A^2\sin^2(h\beta)[\pm(A_3 - A_4) + (3A_6 - A_5)\cos(h\beta)],$$
$$\pm[A_1 + A_4 + A_3\cos(2h\beta)] + (A_2 + A_5)\cos(h\beta) + A_6\cos(h\beta)[4\cos^2(h\beta) - 3] = 0, \tag{25}$$

where the upper (lower) sign corresponds to the nonstaggered (staggered) sine solution.

Using the well-known addition theorem for sine, it is easily shown that the sine solution follows from the two-point quadratic map

$$\phi_{n+1}^2 + \phi_n^2 - 2\phi_{n+1}\phi_n\cos(h\beta) - A^2\sin^2(h\beta) = 0. \tag{26}$$

These solutions are discussed in several special cases in [38].

4. Short-period solutions

We shall now show that apart from the JEF, hyperbolic and trigonometric solutions, there are also several short period and even aperiodic solutions of equation (1) (here we give only some of the solutions that we have obtained, while many more short-period solutions can be found in [38]). In order to obtain these solutions, it is useful to look at the symmetries of equation (1). In particular, note that equation (1) is invariant under $\phi_{n-1} \to \phi_{n+1}$ and $\phi_{n+1} \to \phi_{n-1}$. Further, equation (1) is also invariant under $(\phi_{n-1}, \phi_n, \phi_{n+1}) \to (-\phi_{n-1}, -\phi_n, -\phi_{n+1})$. A consequence of these two symmetries is that if $(\phi_{n-1}, \phi_n, \phi_{n+1})$ is a solution to equation (1) under certain constraints, then $(-\phi_{n-1}, -\phi_n, -\phi_{n+1})$, $(\phi_{n+1}, \phi_n, \phi_{n-1})$ and $(-\phi_{n+1}, -\phi_n, -\phi_{n-1})$ are also solutions of equation (1) provided the same constraints are satisfied.

While obtaining the periodic solutions, the following results, derived by using equations (1) and (2), have been used.

$$\begin{aligned}
&\text{If} \quad \phi_{n-1} = \phi_n = \phi_{n+1} = a, \quad \text{then} \quad a^2 = 1;\\
&\text{If} \quad \phi_{n-1} = \phi_n = -\phi_{n+1} = a, \quad \text{then} \quad \Lambda - 2 = h^2a^2(A_1 + A_3 - A_4);\\
&\text{If} \quad \phi_{n-1} = -\phi_n = \phi_{n+1} = a, \quad \text{then} \quad \Lambda - 4 = h^2a^2(A_1 - A_2 + A_3 + A_4 - A_5 - A_6);\\
&\text{If} \quad \phi_{n-1} = \phi_n = a \quad \text{and} \quad \phi_{n+1} = 0, \quad \text{then} \quad 2(\Lambda-1) = h^2a^2(2A_1 + A_2 + A_3 + A_6);\\
&\text{If} \quad \phi_{n-1} = \phi_{n+1} = a \quad \text{and} \quad \phi_n = 0, \quad \text{then} \quad 2 = h^2a^2(A_5 + A_6);\\
&\text{If} \quad \phi_{n-1} = -\phi_n = a \quad \text{and} \quad \phi_{n+1} = 0, \quad \text{then} \quad 2(\Lambda-3) = h^2a^2(2A_1 - A_2 + A_3 - A_6);\\
&\text{If} \quad \phi_{n-1} = a \quad \text{and} \quad \phi_n = \phi_{n+1} = 0, \quad \text{then} \quad 2 = h^2a^2A_6;\\
&\text{If} \quad \phi_n = a \quad \text{and} \quad \phi_{n-1} = \phi_{n+1} = 0, \quad \text{then} \quad \Lambda - 2 = h^2a^2A_1.
\end{aligned} \tag{27}$$

We mention below all the solutions with periods 2, 3 and 4. Many more solutions with period > 4, can be found in [38] where it has been shown that corresponding to most of these solutions, one also has periodic solutions with an arbitrarily large period as well as *aperiodic* solutions.

For the solutions we first give its number, then the period, the form of the solution, the conditions for the existence of this solution and, possibly, some comments on the solution. All these solutions satisfy constraint (2).

(i) Period 2; $\phi = (\ldots, a, -a, \ldots)$; $\Lambda - 4 = h^2a^2(A_1 - A_2 + A_3 + A_4 - A_5 - A_6)$.

(ii) Period 2; $\phi = (\ldots, a, 0, \ldots)$; $\Lambda - 2 = h^2a^2A_1, 2 = (A_5 + A_6)h^2a^2$.

(iii) Period 2; $\phi = (\ldots, a, 1/a, \ldots)$; $A_1 = A_5 + A_6 = 0, h^2A_2 = \Lambda - 2, h^2(A_4 + A_3) = 2, a$ is an arbitrary real number.

(iv) Period 3; $\phi_n = (a, 0, 0)$; $h^2a^2A_6 = 2$; $h^2a^2A_1 = \Lambda - 2$.

(v) Period 3; $\phi = (\ldots, a, a, 0, \ldots)$; $a^2h^2(A_2 + 2A_1 + A_3 + A_6) = 2(\Lambda - 1), h^2a^2(A_5 + A_6) = 2$.

(vi) Period 3; $\phi = (\ldots, a, -a, 0, \ldots)$; $a^2h^2(2A_1 + A_3 - A_2 - A_6) = 2(\Lambda - 3)$.

(vii) Period 3; $\phi = (\ldots, a, a, -a, \ldots)$; $a^2h^2(A_1 + A_3 - A_4) = \Lambda - 2, a^2h^2(A_1 + A_3 + A_4 - A_2 - A_5 - A_6) = \Lambda - 4$.

(viii) Period 4; $\phi = (\ldots, a, a, a, 0, \ldots)$; $a^2 = 1, h^2(A_2 + 2A_1 + A_3 + A_6) = 2(\Lambda - 1), h^2(A_5 + A_6) = 2$. From here it follows that such a solution is valid provided $A_2 + A_3 + 2A_4 + A_5 = 0$.

(ix) Period 4; $\phi = (\ldots, a, a, a, -a, \ldots)$; $a^2 = 1, h^2(A_1 + A_3 - A_4) = \Lambda - 2, h^2(A_1 + A_3 + A_4 - A_2 - A_5 - A_6) = \Lambda - 4$. In view of constraint (2), such a solution is valid only if $A_4 = 0$.

(x) Period 4; $\phi = (\ldots, a, b, -a, -b, \ldots)$, where $a^2 \neq b^2$; $(a^2 + b^2)h^2A_1 = \Lambda - 2, A_3 = A_1 + A_4, A_1 \neq 0$. Thus one has a one-parameter family of solutions. It is easily seen that such a solution will exist in Model 2 (in case δ is arbitrary but nonzero, while γ is arbitrary), Model 6 (in case $\gamma = 0, \alpha = 1/4$) and Hamiltonian Model 8 (in case $\alpha_1 = \alpha_3$). In the special case of $A_3 = A_4, A_1 = 0$ and $\Lambda = 2$, one, in fact, has a two-parameter family of solutions in the sense that now both a and b are arbitrary real numbers.

(xi) Period 4; $\phi = (\ldots, a, a, -a, -a, \ldots)$; $a^2h^2(A_1 + A_3 - A_4) = \Lambda - 2$.

(xii) Period 4; $\phi = (\ldots, a, 0, -a, 0, \ldots)$; $a^2h^2A_1 = \Lambda - 2$.

(xiii) Period 4; $\phi = (\ldots, a, 0, -a, a, \ldots)$; $a^2h^2(A_1 + A_3 - A_4) = \Lambda - 2, a^2h^2(2A_1 + A_3 + A_2 + A_6) = 2(\Lambda - 1), a^2h^2(2A_1 + A_3 - A_2 - A_6) = 2(\Lambda - 3)$. From here it follows that such a solution is valid provided $A_3 = 2A_4$.

(xiv) Period 4; $\phi = (\ldots, a, a, 0, 0, \ldots)$; $h^2a^2A_6 = 2$; $h^2a^2(2A_1 + A_2 + A_3 + A_6) = 2(\Lambda - 1)$.

(xv) Period 4; $\phi = (\ldots, a, -a, 0, 0, \ldots)$; $h^2a^2A_6 = 2$; $h^2a^2(2A_1 - A_2 + A_3 - A_6) = 2(\Lambda - 3)$.

5. Two-point maps

5.1. General maps that include the integration constant

As pointed out in section 1, Model 2, for any value of δ and γ, has the first integral with the integration constant C expressed by equation (6).

In case only A_2 and A_4 are nonzero (i.e., Model 7 at $\sigma = 0$), the discrete model equation (1) has the first integral with the integration constant C constructed in [28]. Here we

present it in the following form [18]:

$$W(\phi_n, \phi_{n+1}) \equiv \phi_n^2 + \phi_{n+1}^2 - \frac{Y}{2-\Lambda}\phi_n^2\phi_{n+1}^2 - 2Z\phi_n\phi_{n+1} - \frac{CY}{2-\Lambda} = 0,$$
$$Z = \frac{(2-\Lambda)^2 - Ch^4A_4^2}{2\,(2-\Lambda) + Ch^4A_2A_4}, \qquad Y = h^2(A_4 + A_2Z), \tag{28}$$

which is precisely of the Quispel–Roberts–Thompson (QRT) form as given in [27]. Note that equation (1) with only A_2 and A_4 nonzero can be expressed in terms of equation (28) as follows:

$$\ddot{\phi}_n = \frac{2-\Lambda}{2Z(\phi_{n+1} - \phi_{n-1})}\Big\{W(\phi_n, \phi_{n+1}) - W(\phi_{n-1}, \phi_n) + \frac{h^2A_4}{2-\Lambda}\left[\phi_{n+1}^2 W(\phi_{n-1}, \phi_n) - \phi_{n-1}^2 W(\phi_n, \phi_{n+1})\right]\Big\}. \tag{29}$$

It is now clear that the static solutions to equation (1) with only A_2 and A_4 nonzero can be found from the two-point map $W(\phi_n, \phi_{n+1}) = 0$. Note that for small h one has $Z \approx (2-\Lambda)/2$ and the last term in the curly bracket of equation (29) can be neglected and one obtains an equation similar to equation (8) of Model 2. In other words, the model with only A_2 and A_4 nonzero can be regarded as the Model 2 modified by the O-term (the last term in the curly bracket), i.e. the term which disappears in the continuum limit and vanishes upon substituting $W(\phi_n, \phi_{n+1}) = 0$ [16]. In the continuum limit, the DFI equation (28) reduces to the first integral of the static continuum ϕ^4, equation (4).

We have checked that from the two-point map $W(\phi_n, \phi_{n+1}) = 0$ one can obtain the staggered as well as the nonstaggered JEF solutions sn, cn and dn derived in section 3 from the three-point static equation (1) in case only A_2 and A_4 are nonzero.

5.2. Two-point maps for the Models 9–12

What about a universal map for the Models 9–12? Unfortunately, so far we have not been able to find one for any of these cases. However, as we now show, corresponding to any static JEF solution, we can always generate a map. In particular, we now obtain a map corresponding to the sn JEF solution while similar maps for the cn and dn solutions can be found in [38]. We will also demonstrate that the corresponding map can always be factorized and reduced to the QRT form [27].

The exact sn solution for model equation (1) has the form of equation (13) with $f = \mathrm{sn}$. On using the well-known identity

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}(u)\,\mathrm{cn}(v)\,\mathrm{dn}(v) + \mathrm{sn}(v)\,\mathrm{cn}(u)\,\mathrm{dn}(u)}{1 - m\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)}, \tag{30}$$

it immediately follows that ϕ_{n+1} and ϕ_n are related by the map

$$\phi_{n+1} = \frac{\phi_n\,\mathrm{cn}(h\beta)\,\mathrm{dn}(h\beta) + \mathrm{sn}(h\beta)\sqrt{A^2 - (1+m)\phi_n^2 + m\left(\phi_n^4/A^2\right)}}{1 - m\,\mathrm{sn}^2(h\beta)\phi_n^2/A^2}. \tag{31}$$

This can be simplified and put in the following factorized form:

$$\left[A^2 - m\,\mathrm{sn}^2(h\beta)\phi_n^2\right]W_{\mathrm{sn}}(\phi_n, \phi_{n+1}) = 0, \tag{32}$$

where

$$W_{\mathrm{sn}}(\phi_n, \phi_{n+1}) = A^2\left(\phi_n^2 + \phi_{n+1}^2\right) - m\,\mathrm{sn}^2(h\beta)\phi_n^2\phi_{n+1}^2 - 2A^2\,\mathrm{cn}(h\beta)\,\mathrm{dn}(h\beta)\phi_n\phi_{n+1} - A^4\,\mathrm{sn}^2(h\beta). \tag{33}$$

Since the vanishing of the first bracket in equation (32) is a trivial possibility, effectively the map in this case is given by $W_{\mathrm{sn}}(\phi_n, \phi_{n+1}) = 0$ which is precisely of the QRT form [27].

5.3. Particular factorized static problems

In few cases, the static problem can be factorized and one can obtain some of the exact solutions, such as those presented in section 4, from this lower order algebraic equation. As an illustration, we now present a few examples of such factorized static problems.

As our first example, we note that in case $\gamma = 0$, $\delta = 1/4$, and for the integration constant chosen as $Ch^2 = (\Lambda - 4)^2/(2\Lambda)$, the two-point map for the Model 2, given by equation (6), can be factorized as

$$-\frac{4}{\Lambda}\left(1 - \frac{\Lambda}{4}\phi_n\phi_{n+1}\right)\left[\Lambda - 2 - \frac{\Lambda}{4}\left(\phi_n^2 + \phi_{n+1}^2\right)\right] = 0. \tag{34}$$

It is easily checked that the last multiplier of equation (34) generates the four-periodic solution $\phi_n \equiv (\ldots, a, b, -a, -b, \ldots)$ with $\Lambda(a^2 + b^2) = 4(\Lambda - 2)$ (solution (x) of section 4). This solution is a TI solution because it can also be written in the form of the four-periodic sine solution as given by equation (14) with $g = \sin$ and with parameters satisfying equation (25).

As a second example we take the Bender–Tovbis model, i.e. the Model 2 with $\gamma = \delta = 0$. In this case, the two-point map equation (6), with $\Lambda = 2$ and $C = 1$, can be factorized as

$$\left(1 - \phi_{n-1}^2\right)\left(\phi_n^2 - 1\right) = 0. \tag{35}$$

Now observe that any sequence of ± 1 satisfies equation (35) and hence the Bender–Tovbis model at $\Lambda = 2$ and $C = 1$. To obtain this aperiodic solution one can use either of the multipliers of equation (35). This is possible because the solutions are derived from the two-point rather than the three-point map. This example explains how one can obtain aperiodic solutions from the factorized maps.

In the above examples the short-periodic solutions were obtained from the factorized two-point map. More examples of this sort can be found in [18] where short-periodic solutions are derived from the two-point map, equation (28), in case $\Lambda = 2$ and either only A_2 or only A_4 is nonzero.

As another example, the SW Model 5 can be written in the form [16]

$$\ddot{\phi}_n = -v(\phi_{n-1}, \phi_n)\frac{\partial}{\partial \phi_n}v(\phi_{n-1}, \phi_n) - v(\phi_n, \phi_{n+1})\frac{\partial}{\partial \phi_n}v(\phi_n, \phi_{n+1}), \tag{36}$$

where $v(\phi_{n-1}, \phi_n)$ is given by equation (11). It is clear that the equation $v(\phi_{n-1}, \phi_n) = 0$ generates static kink and inverted kink solutions for this model.

The following two examples are interesting because they give the short-periodic solutions, from a set of *two* finite-difference equations, rather than from a two-point map.

We note that the SW Model 5 can also be written in the form

$$\begin{aligned}\ddot{\phi}_n = {} & \frac{1}{h^2}(\phi_{n+1} + \phi_{n-1} - 2\phi_n) + \lambda\phi_n - \frac{\lambda}{18}(\phi_{n-1} + 2\phi_n)\left(\phi_{n-1}^2 + \phi_{n-1}\phi_n + \phi_n^2\right) \\ & - \frac{\lambda}{18}(\phi_{n+1} + 2\phi_n)\left(\phi_n^2 + \phi_n\phi_{n+1} + \phi_{n+1}^2\right).\end{aligned} \tag{37}$$

We now observe that in case the following two-point equation holds

$$\phi_{n-1}^2 + \phi_{n-1}\phi_n + \phi_n^2 = 6(\Lambda - 3)/\Lambda, \tag{38}$$

then the static version of equation (37) reduces to

$$(\Lambda - 6)(\phi_{n-1} + \phi_n + \phi_{n+1}) = 0. \tag{39}$$

It is easily checked that the two-point map equation (38) generates the exact three-periodic sin as well as the six-periodic staggered sin solutions to the SW Model 5 (see equation (14) with $g = \sin$ and with parameters satisfying equation (25)) and for these solutions, equation (39) is also satisfied. Further, equations (38) and (39) for the case of $\Lambda = 6$ also generate the short

periodic solutions (ii) and (iv), as given in section 4, and some other solutions with a longer period [38].

Similarly, the equation of motion for the Hamiltonian Model 8 can be written in the form

$$\ddot{\phi}_n = \frac{1}{h^2}(\phi_{n+1} + \phi_{n-1} - 2\phi_n) + \lambda\phi_n - \lambda(\alpha_2\phi_{n-1} + 2\alpha_1\phi_n)\left(\phi_{n-1}^2 + \frac{\alpha_2}{\alpha_1}\phi_{n-1}\phi_n + \phi_n^2\right) - \lambda(\alpha_2\phi_{n+1} + 2\alpha_1\phi_n)\left(\phi_{n+1}^2 + \frac{\alpha_2}{\alpha_1}\phi_{n+1}\phi_n + \phi_n^2\right). \tag{40}$$

In case the following two-point equation holds

$$\phi_{n-1}^2 + (\alpha_2/\alpha_1)\phi_{n-1}\phi_n + \phi_n^2 = H, \tag{41}$$

then the static version of equation (40) reduces to

$$(\phi_{n-1} + \phi_{n+1})(1 - \Lambda H\alpha_2) + (\Lambda - 2 - 4\alpha_1\Lambda H)\phi_n = 0, \qquad H = \frac{\Lambda - 2 - \alpha_2/\alpha_1}{\Lambda\left(4\alpha_1 - \alpha_2^2/\alpha_1\right)}, \tag{42}$$

provided

$$\alpha_3 = \alpha_1 + \alpha_2^2/(2\alpha_1). \tag{43}$$

In the special case of $\alpha_1 = \alpha_2 = 1/18$, this reduces to the SW model. The two-point map equation (41) generates the exact sine as well as the staggered sine solutions to the Hamiltonian Model 8 with $\cos(h\beta) = \mp\alpha_2/(2\alpha_1)$, and for these solutions equation (42) is also satisfied.

It is interesting to note that for the factorized equations (38) and (39), and also for the factorized equations (41) and (42), one has to satisfy two lower-order finite-difference equations *simultaneously*, one of those equations is a two-point one, while another is a three-point one.

None of the factorized problems discussed in this section contain the integration constant and thus they generate only particular solutions. Some of them are TI solutions, for example, the four-periodic solution derivable from equation (34), while others are not, for example, an arbitrary sequence of ± 1, derivable from equation (35).

Solutions constructed in this section from factorized problems do not survive in the continuum limit because the factorized problems do not reduce to equation (3) or (4) in the continuum limit.

6. Goldstone modes

6.1. Goldstone mode of a TI static solution

Let ϕ_n^0 be a static solution to equation (1). To study the dynamics in the vicinity of this solution we substitute the ansatz $\phi_n(t) = \phi_n^0 + \varepsilon_n(t)$ into equation (1), and obtain the following linearized equation:

$$\ddot{\varepsilon}_n = K_{n,n-1}\varepsilon_{n-1} + K_{n,n}\varepsilon_n + K_{n,n+1}\varepsilon_{n+1}, \tag{44}$$

with

$$K_{n,n-1} = \frac{1}{h^2} - \frac{A_2}{2}\left(\phi_n^0\right)^2 - A_3\phi_{n-1}^0\phi_n^0 - A_4\phi_n^0\phi_{n+1}^0 - \frac{A_5}{2}\phi_{n+1}^0\left(2\phi_{n-1}^0 + \phi_{n+1}^0\right) - \frac{3A_6}{2}\left(\phi_{n-1}^0\right)^2,$$
$$K_{n,n} = \lambda - \frac{2}{h^2} - 3A_1\left(\phi_n^0\right)^2 - A_2\phi_n^0\left(\phi_{n-1}^0 + \phi_{n+1}^0\right) - \frac{A_3}{2}\left[\left(\phi_{n-1}^0\right)^2 + \left(\phi_{n+1}^0\right)^2\right] - A_4\phi_{n-1}^0\phi_{n+1}^0,$$
$$K_{n,n+1} = \frac{1}{h^2} - \frac{A_2}{2}\left(\phi_n^0\right)^2 - A_3\phi_n^0\phi_{n+1}^0 - A_4\phi_{n-1}^0\phi_n^0 - \frac{A_5}{2}\phi_{n-1}^0\left(\phi_{n-1}^0 + 2\phi_{n+1}^0\right) - \frac{3A_6}{2}\left(\phi_{n+1}^0\right)^2. \tag{45}$$

Looking for solutions of equation (44) of the form $\varepsilon_n(t) = U_n \exp(\pm i\omega t)$, we come to the eigenvalue problem

$$[K]\,\mathbf{U} = -\omega^2 \mathbf{U}, \tag{46}$$

where vector $\mathbf{U}$ contains U_n and the nonzero coefficients of matrix $[K]$ are given by equation (45).

If ϕ_n^0 is a TI static solution then it can be shifted along the chain by an arbitrary x_0, $\phi_n^0 = \phi(n + x_0)$. The eigenvector corresponding to the zero-frequency translational Goldstone mode, $\mathbf{U}_G$, has components ϕ_n', where prime means derivative of ϕ with respect to its argument. To confirm that, we substitute $\mathbf{U}_G = \{\phi_n'\}$ into equation (46) with $\omega^2 = 0$ and obtain

$$K_{n,n-1}\phi_{n-1}' + K_{n,n}\phi_n' + K_{n,n+1}\phi_{n+1}' = 0. \tag{47}$$

The last expression is an identity because it coincides with the derivative of static version of equation (1) with respect to x_0, and such a derivation is possible for the TI static solution, which is an equilibrium solution for any x_0. We thus have proved that any TI static solution has the zero-frequency translational mode $\mathbf{U}_G = \{\phi_n'\}$. Particularly, for any static JEF, hyperbolic or trigonometric function solution given in section 3, one can easily find the corresponding TI mode as it is proportional to the derivative of the solution with respect to its argument.

Looking for solutions of equations (44) and (45) with $\phi_n^0 = 1$ of the form of small-amplitude phonons, $\varepsilon_n(t) \sim \exp(ikn \pm i\omega t)$, where k denotes wave number and ω is frequency, one obtains the spectrum of the vacuum for the discrete model of equation (1),

$$\omega^2 = 2\lambda + 2\left[\frac{2}{h^2} - A_2 - 2A_3 - 2A_4 - 3A_5 - 3A_6\right]\sin^2\left(\frac{k}{2}\right). \tag{48}$$

Another vacuum solution, $\phi_n = 0$, also supports phonons with the dispersion relation

$$\omega^2 = -\lambda + (4/h^2)\sin^2(k/2), \tag{49}$$

and this vacuum solution is stable provided $\lambda < 0$.

6.2. Goldstone modes of some short-period static solutions

If a static solution does not possess the zero-frequency translational Goldstone mode, then this solution is not a TI one. The opposite, in general is not true, i.e., a particular static solution may have the Goldstone mode only at certain positions with respect to the lattice x_0, but a TI solution must have such a mode at any x_0. It is insightful to check whether some of the short-periodic solutions derived in section 4 have the Goldstone mode.

Three periodic solution to the SW Model 5, as found in section 3.2, is $\phi_n = A\sin[(2\pi/3)(n + x_0)]$ with $A^2 = 8(\Lambda - 3)/\Lambda$. The solution includes an arbitrary shift x_0 and thus is a TI solution with the Goldstone mode described in section 6.1.

Four periodic solution $(\ldots, a, b, -a, -b, \ldots)$, i.e., solution (x) of section 4, exists in case $A_1 \neq 0$ under the constraint $A_1 - A_3 + A_4 = 0$ and has $a^2 + b^2 = (\Lambda - 2)/(A_1 h^2)$. This one-parameter solution is a TI solution because it is also the sine solution given in section 3.2. In particular, it can be written in the form $\phi_n = A\sin[\pi(n + x_0)/2]$ with an arbitrary shift x_0 and $A^2 = (\Lambda - 2)/(A_1 h^2)$. Being a TI solution, it possesses the Goldstone mode at any x_0, as was demonstrated in section 6.1. Note that Models 2, 6, 8 and SW Model 5 have $A_1 \neq 0$ but the constraint $A_1 - A_3 + A_4 = 0$ is not satisfied for the SW Model 5, while it is satisfied for Model 2 (at arbitrary γ and arbitrary nonzero δ), Model 6 (at $\gamma = 0, \alpha = 1/4$) and

Model 8 (at $\alpha_1 = \alpha_3$). Thus, while Models 2, 6 and 8 support this TI four-periodic solution, Model 5 does not.

Four periodic solution of the form $(\ldots, a, 0, -a, 0, \ldots)$, as found in section 4, exists for $A_1 \neq 0$ and has $a^2 = (\Lambda - 2)/(A_1 h^2)$. Inserting this solution into equation (46) one finds

$$\begin{bmatrix} \alpha & \beta & 0 & \beta \\ \delta & \gamma & \delta & 0 \\ 0 & \beta & \alpha & \beta \\ \delta & 0 & \delta & \gamma \end{bmatrix} \begin{Bmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{Bmatrix} = -\omega^2 \begin{Bmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{Bmatrix}, \tag{50}$$

where

$$\begin{aligned} &\alpha = -2\frac{\Lambda - 2}{h^2}, \qquad \beta = \frac{1}{h^2} - A_2 \frac{\Lambda - 2}{2A_1 h^2}, \\ &\gamma = (A_1 - A_3 + A_4)\frac{\Lambda - 2}{A_1 h^2}, \qquad \delta = \frac{1}{h^2} + (A_5 - 3A_6)\frac{\Lambda - 2}{2A_1 h^2}. \end{aligned} \tag{51}$$

The characteristic relation corresponding to equation (50) is $(\alpha + \omega^2)(\gamma + \omega^2)$ $[(\alpha + \omega^2)(\gamma + \omega^2) - 4\beta\delta] = 0$ and zero-frequency modes are possible when any of the following three conditions, $\alpha = 0, \gamma = 0, \alpha\gamma - 4\beta\delta = 0$, is satisfied. The considered four-periodic solution can also be expressed as $\phi_n = A\cos(\pi n/2)$ with $A^2 = (\Lambda - 2)/(A_1 h^2)$. The expected Goldstone mode is proportional to the derivative of this function with respect to its argument, and thus, should have the form $U_0 = U_2 = 0, U_1 = -U_3 = k$ with an arbitrary $k \neq 0$. Substituting this into equation (50) with $\omega^2 = 0$ one finds that the Goldstone mode corresponds to $\gamma = 0$. Taking into account equation (51), we conclude that the considered four-periodic solution possesses the Goldstone mode under the condition

$$A_1 - A_3 + A_4 = 0. \tag{52}$$

It is interesting to note that while the SW Model 5 supports the considered four-periodic solution, but it does not satisfy the condition of equation (52) so that the solution is *not* a TI one. On the other hand, Model 2 supports the considered four-periodic solution and also the condition of equation (52) is satisfied, so that the Model 2 supports this four-periodic TI solution.

In [38], Goldstone modes for some other short-periodic solutions are analyzed.

7. Numerical results

7.1. Five-periodic static solution in the Speight and Ward Model 5

Here we address the problem of integrability of the SW Model 5. As mentioned earlier, in this model while there is a two-point map equation (11) to derive the kink solution, but a general two-point map, that includes the integration constant, is not known. We will give numerical evidence that the Model 5 is not integrable and the two-point map for obtaining a two-parameter set of static solutions cannot be constructed. For this purpose we try to construct a static five-periodic solution to this model and show below that it can be constructed only in the case of highly symmetric positions of the solution with respect to the lattice so that this solution is not a TI one and hence possesses the Peierls–Nabarro potential. We have chosen the five-periodic solution for this study because it is relatively simple and it corresponds to a non-factorized static problem. Simple short-periodic solutions described in section 4 are not suitable for this study because they are obtained from low-order algebraic equations, i.e., from factorized static problems that do not represent the model in its full generality.

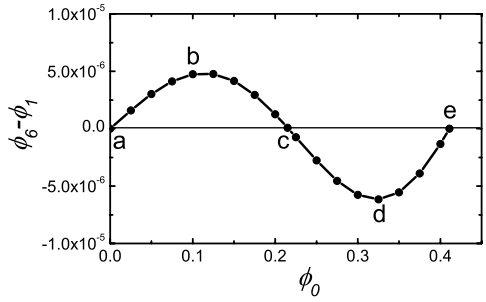

Figure 1. Difference between ϕ_6 and ϕ_1 as a function of ϕ_0 for the static solutions constructed for the Speight and Ward Model 5 from the three-point map $\phi_{n+1} = f(\phi_{n-1}, \phi_n)$ for chosen ϕ_0 and numerically found ϕ_1 such that $\phi_5 = \phi_0$ (half-period is shown). If the three-point problem is reducible to a two-point problem $\phi_{n+1} = g(\phi_n)$ then from having $\phi_5 = \phi_0$ one must also have $\phi_6 = \phi_1$, which is not the case and thus, for this solution the two-point reduction is impossible. Static solutions generated by the three-point map for the points marked with a to e are shown in the corresponding panels of figure 2.

We set for the model parameters $\lambda = 1$, $h = 1.3$ and A_k corresponding to the Model 5 as given in the Introduction. The three-point static problem of equation (1) is written in the form of the map $\phi_{n+1} = f(\phi_{n-1}, \phi_n)$ which, for given ϕ_0 and ϕ_1, generates a static solution. For chosen ϕ_0 we numerically find ϕ_1 such that in the iteratively obtained solution $\phi_5 = \phi_0$, and check whether in this case we also have $\phi_6 = \phi_1$. If the three-point map $\phi_{n+1} = f(\phi_{n-1}, \phi_n)$ is reducible to a two-point map $\phi_{n+1} = g(\phi_n)$ then having $\phi_5 = \phi_0$ we must also have $\phi_6 = \phi_1$. However, as it can be seen from figure 1, $\phi_6 - \phi_1$ is equal to zero only for a discrete set of ϕ_0. We note that the result presented in figure 1 is not a numerical artifact. We have done the simulations with single and double precision and obtained the results indistinguishable on the scale of figure 1. In figures 2(*a*)–(*e*) we plot the structures generated by the three-point map for different ϕ_0 indicated in figure 1 by, correspondingly, letters a–e. Solutions in (*b*) and (*d*) are *not* five-periodic solutions because ϕ_6 differs from ϕ_1 by the amount shown in figure 1. These solutions are modulated five-periodic structures but this cannot be seen in figure 2 because the period of the modulated structure is very large.

Static solutions shown in figures 2(*a*) and (*c*) are the five-periodic equilibrium solutions for which the small-amplitude vibrational spectrum can be calculated as described in section 6.1. We find that these five-periodic structures do not possess the zero-frequency Goldstone mode, but they have a nearly translational mode with frequency $\omega = 0.0036$ for the structure in figure 2(*a*) and a purely imaginary frequency $\omega = 0.0036\mathrm{i}$ for the structure in figure 2(*c*). Thus, the five-periodic structure in the SW Model 5 is not a TI one and it experiences the PN potential with a minimum energy corresponding to the structure in (*a*) and a maximum energy corresponding to the structure in (*c*).

We have studied some other periodic solutions, for example, seven- and eight-periodic ones and have obtained results qualitatively similar to that for the five-periodic structure. We conclude that the static solutions supported by the SW Model 5, except for the kinks, anti-kinks, sine and staggered-sine solutions, usually have the PN potential. The corresponding three-point static problem is non-integrable and cannot be reduced to a two-point problem, again, except for some particular solutions.

This result is not surprising at all and it could be expected taking into account that the derivation of the two-point map equation (11), from which the kink solution can be derived, was done for the integration constant $C = 0$ in equation (4). Only for this particular value

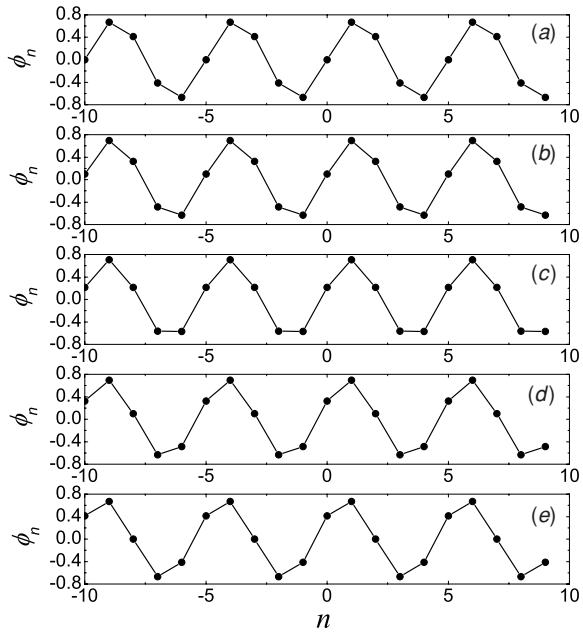

Figure 2. Static solutions for the Speight and Ward Model 5 generated from the three-point map $\phi_{n+1} = f(\phi_{n-1}, \phi_n)$ for chosen ϕ_0 and numerically found ϕ_1 such that $\phi_5 = \phi_0$. Panels (*a*) to (*e*) show the results for the values of ϕ_0 marked with a to e in figure 1. Solutions in (*b*) and (*d*) are *not* five-periodic solutions because ϕ_6 differs from ϕ_1 by the amount shown in figure 1. Highly symmetric solutions in (*a*), (*c*) and (*e*) are the five-periodic ones and for them $\phi_6 = \phi_1$, as it can be seen from figure 1. Solutions in (*a*) and (*e*) correspond to a minimum of the Peierls–Nabarro potential while that in (*c*) to a maximum of this potential.

of the integration constant, the resulting discrete model supports the TI solutions and those solutions are kinks. A Hamiltonian ED model of the Klein–Gordon type that generalizes the SW model has been derived in section IIC of [16]. The model includes the integration constant and thus supports a two-parameter set of TI static solutions, although it is rather complex even for the cubic nonlinearity.

7.2. Static kinks

Here, after a brief discussion about the JEF solutions, we focus on the analysis of the kink solutions because they are discussed in applications more often than the periodic solutions.

As was mentioned earlier, the JEF solutions and their hyperbolic function limit solutions such as kink and pulse exist in the model equation (1) in the seven cases, of which the first three cases with (i) only A_2 nonzero, (ii) only A_4 nonzero and (iii) only A_2 and A_4 nonzero are qualitatively different from the other four cases discussed in section 3. In particular, while for the first three cases one has two conditions for finding the JEF solution parameters A and β, in the remaining four cases one has to satisfy one more condition. This additional constraint couples the model parameters A_k to the lattice spacing h. As a result, in the last four cases, for fixed A_k, one has TI solutions only at particular h, while in the first three cases, even for fixed A_k, one has TI solutions for any h.

Let us demonstrate this qualitative difference between two groups of models by comparison of the properties of the static kinks. The first group of models will be represented

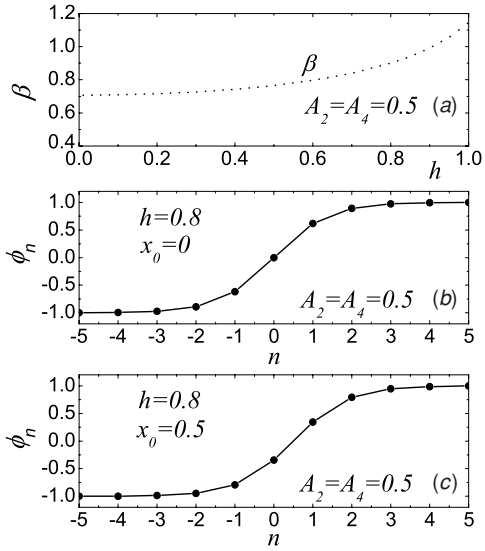

Figure 3. Static kink in case (iii) with only A_2 and A_4 nonzero: (a) the kink parameter β as a function of h; (b) on-site kink at $h = 0.8$; (c) inter-site kink at $h = 0.8$. This model admits TI static solutions, including the kink solution, at constant (h-independent) model parameters A_k, as can be seen in (a).

by case (iii) with only A_2 and A_4 nonzero (i.e. Model 7 at $\sigma = 0$), while from the four models of the second group we will choose the case of Model 9 with only A_3 and A_5 nonzero, and Model 11 with only A_3, A_4 and A_5 nonzero. For the kink solutions discussed below we will always set $\lambda = 1$.

Kink in case (iii) with only A_2 and A_4 nonzero. In this case, the parameters of the kink solution, equation (14) with $g = \tanh$, $S = 1$, are given by equation (22). For given model parameters h and A_4 one can find the inverse kink width β by solving the second equation in equation (22). The model parameter A_2 must satisfy the continuity constraint given by the last expression in equation (22). A particular feature of this discrete ϕ^4 model is that it admits the TI solutions at constant (h-independent) A_k, see figure 3.

Vibrational spectrum of the lattice containing a kink at different positions with respect to the lattice x_0 is shown in figure 4 for model parameters $h = 0.8$, $A_2 = A_4 = 0.5$, $\lambda = 1$. The corresponding kink profiles at $x_0 = 0$ and $x_0 = 0.5$ are shown in figures 3(b) and (c), respectively. At any position x_0 the kink possesses the zero-frequency Goldstone translational mode. Straight horizontal line at $\omega = \sqrt{2}$ shows the lower bound of the phonon band, see equation (48).

Kink in Model 9 with only A_3 and A_5 nonzero. Model parameters and kink parameters in this case are given by equation (23). For chosen A_3 (or A_5) one can find A_5 (or A_3) from the continuity constraint (equation (23)) and then find the inverse kink width β solving the second equation in equation (23). Finally, one of equation (23) relates the model parameters A_k to the lattice spacing h. In figure 5, we show (a) the model parameters A_k and kink inverse width β as functions of h. In the region of lattice parameter around $h = 0.5$ it is possible to have two different static kink solutions at the same h which is illustrated in (b) and (c). In both cases $h = 0.5$, but model parameters A_k and the inverse kink width β are different (shown in each

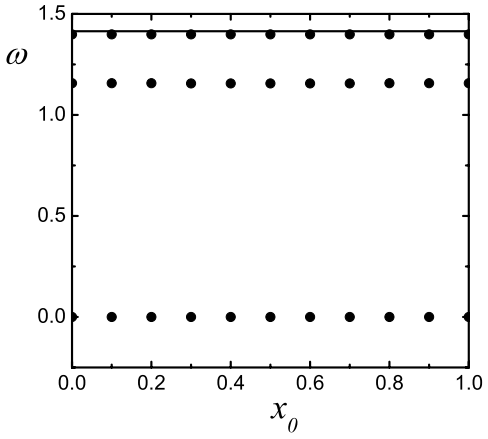

Figure 4. Spectrum of the lattice with a kink in case (iii) with only A_2 and A_4 nonzero. Straight horizontal line at $\omega = \sqrt{2}$ shows the lower bound of the phonon band while dots show the kink's internal modes calculated for the kink at various positions x_0 with respect to the lattice. At any position x_0 the kink possesses the zero-frequency Goldstone translational mode. The kinks at $x_0 = 0$ and $x_0 = 0.5$ are shown in figures 3(*b*) and (*c*), respectively. Model parameters: $h = 0.8$, $A_2 = A_4 = 0.5$, $\lambda = 1$.

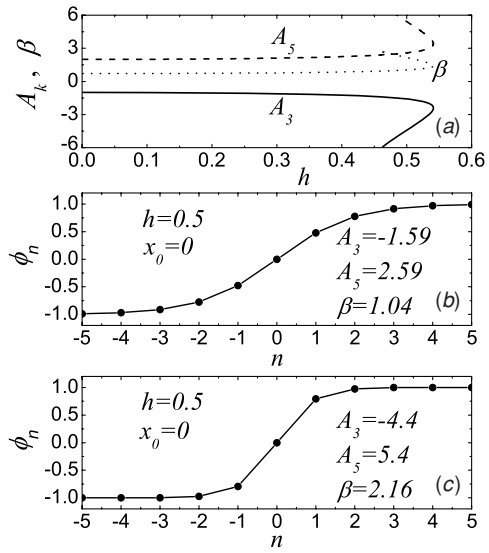

Figure 5. Static kink in Model 9 with only A_3 and A_5 nonzero. (*a*) Model parameters A_k and kink inverse width β as functions of h. In the region of lattice parameter around $h = 0.5$ it is possible to have two different static kink solutions at the same h which is illustrated in (*b*) and (*c*). In both cases $h = 0.5$, but model parameters A_k and the inverse kink width β are different (shown in each panel). Both kinks are stable and have zero-frequency translational Goldstone mode at any position with respect to the lattice x_0.

panel). Both kinks are stable and have a zero-frequency translational Goldstone mode at any position with respect to the lattice x_0.

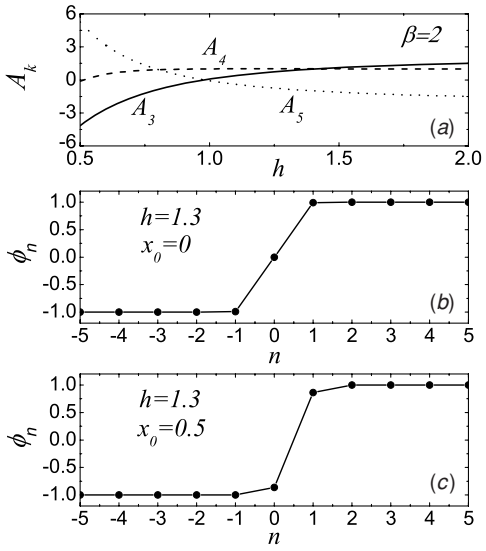

Figure 6. Static kink in Model 11 with only A_3, A_4 and A_5 nonzero. (*a*) Model parameters A_k as functions of h at *fixed* inverse kink width $\beta = 2$. (*b*) On-site kink at $h = 1.3$. (*c*) Inter-site kink at $h = 1.3$. Other model parameters for (*b*) and (*c*) are $A_3 = 0.8297$, $A_4 = 1.0092$, $A_5 = -0.8389$ and $\lambda = 1$.

Kink in Model 11 with only A_3, A_4 and A_5 nonzero. Kink and model parameters are related by equation (24). In figure 6, we plot (*a*) model parameters A_k as functions of h at *fixed* inverse kink width $\beta = 2$; (*b*) the on-site kink at $h = 1.3$; and (*c*) the inter-site kink at $h = 1.3$. Other model parameters for (*b*) and (*c*) are $A_3 = 0.8297$, $A_4 = 1.0092$, $A_5 = -0.8389$ and $\lambda = 1$.

Note that in figure 6(*a*) all A_k vary with h in a wide range of lattice spacing h but, interestingly, the inverse kink width β is constant ($= 2$). In the classical discrete ϕ^4 model and in the models with h-independent parameters A_k the kink width usually decreases with increase in h. On the other hand, it is possible to get from equation (24) one constant model parameter, with two other model parameters A_k and the kink parameter β being functions of h.

8. Discussion and conclusions

In this paper, we have introduced a rather general discrete ϕ^4 model of which all known models in the literature are special cases. We could find seven special cases when the model as given by equation (1) supports the exact static JEF and hence hyperbolic kink and pulse solutions. Two of those seven cases have been analyzed in [16, 18] (and also in earlier works, e.g., [36]), while for the remaining five cases, JEF solutions were given in section 3.

The exact solutions constructed for the considered discrete ϕ^4 model are important for the theory of the ED lattices. Indeed, the JEF static solutions with an arbitrary shift along the lattice x_0 are the TI solutions with the zero-frequency Goldstone mode, i.e., solutions that are free of the PN potential [16]. The four new ED ϕ^4 models derived in this study are Model 9 with only A_3 and A_5 nonzero; Model 10 with only A_2, A_3 and A_5 nonzero; Model 11 with only A_3, A_4 and A_5 nonzero; and Model 12 with only A_2, A_3, A_4 and A_5 nonzero. Each of these models (like the other three, i.e. Model 2 at $\gamma = \delta = 0$, Models 4 and 7 at $\sigma = 0$)

supports a two-dimensional set of TI static solutions that can be parameterized by the points of the plane (m, x_0). For *fixed* model parameters A_k these models support TI solutions only for a particular lattice spacing h. Note, however, that the other three models, i.e. Model 2 at $\gamma = \delta = 0$, Models 4 and 7 at $\sigma = 0$, for fixed model parameters A_2 and A_4, support the TI solutions for any value of h. Some special cases of the four new nonlinearities have been reported in [13] and for those nonlinearities the kink solutions have been obtained there.

We also showed that the general model, equation (1), supports periodic sine and staggered sine solutions (such solutions, as it was already mentioned, have been discussed in the literature for various lattice models [30–34]). Remarkably, almost all the known models (even those not supporting the JEF solutions) were found to support these solutions. Besides, a large number of exact, short-periodic and aperiodic static solutions admitted by equation (1) were obtained in section 4, where we gave only a part of the solutions that we have obtained, while many more solutions of this kind can be found in an extended version of this work [38]. While we do not have a rigorous proof, but the few examples discussed in section 5.3 suggest that very likely, the short-periodic, aperiodic as well as trigonometric solutions, in fact follow from low-order algebraic equations. In this context, it is worth pointing out that the sine solution does not follow from the map for Model 2 as well as the map from the Model 7 at $\sigma = 0$ when only A_2 and A_4 are nonzero. The fact that the derived sine solutions do *not* follow from the JEF solutions in the limit $m \to 0$ is in line with the claim that the sine solitons follow not from the full three-point problem (as JEF solutions do) but from particular factorized problems. The factorization can also easily explain the appearance of the aperiodic solutions that can be regarded as the solutions obtained from different multipliers and linked together, as exemplified by the discussion below equation (35).

None of the factorized problems discussed in section 5.3 contain an integration constant and thus they generate only particular solutions. Some of them are TI solutions, for example, the three-periodic solution to the SW Model 5 derivable from equations (38) and (39), while others are not, for example, arbitrary sequence of ± 1, derivable from equation (35). We also discussed several examples in which the three-point problem can be reduced to a set of two lower order finite-difference equations, and one of those equations is a two-point one while another is a three-point one.

The short-periodic solutions and, more generally, the solutions derived from factorized problems very often do not survive the continuum limit because factorized equations usually have a different continuum limit than the original, non-factorized one. In this context we note that sine is not a solution of the continuum ϕ^4 field equation. One exception to this rule is the kink solution to the SW Model 5 for which the reduced two-point problem equation (11) in the continuum limit attains a form which is equivalent to the first integral of the static ϕ^4 field, equation (4).

Coming back to the exact JEF solutions, we emphasize that they are important because by using them one can construct the corresponding two-point maps from which the corresponding solutions can be obtained iteratively. In some cases, the map obtained for a particular JEF solution can be transformed to a general map from which majority of static solutions including other JEF solutions admitted by the model can be constructed. Following this way we could construct the map of equation (28) from which any static solution of equation (1) with only A_2 and A_4 nonzero can be constructed (except for the solutions that result from specially factorized problems). On the other hand, for Models 9–12, while one can obtain a map from a JEF solution, so far we are unable to obtain a general universal map valid for all the JEF solutions.

In section 7.1, we provided numerical evidence that the SW Model 5 does not support TI static solutions other than those derivable from reduced lower order algebraic problems as

discussed in section 5.3. In addition to the well-known TI kink solution, we have found the TI sine and staggered sine solutions to this model. We thus believe that the static SW model is not integrable and a two-point map that includes the integration constant as a parameter cannot be constructed for this model.

Based on the results of the present study one can separate the ED models into two classes. In the first class (ED I) belong the models that support a two-dimensional space of TI static solutions. These solutions, if they are derivable from a two-point nonlinear map, can be parameterized by the points of the plane (C, ϕ_0), where C is the integration constant that can vary continuously within a certain range and ϕ_0 is the initial value of the map that can also vary continuously. Alternatively, if the JEF solutions are known, then the solutions of ED I models can be parameterized by the points of the plane (m, x_0) so that m plays the role of the integration constant C while variation of x_0 plays the role of ϕ_0, and results in the shift of the solution along the lattice. The second class (ED II) is formed by the models that admit TI static solutions with an arbitrary shift along the lattice (controlled by either x_0 or ϕ_0) but corresponding solutions do not include the integration constant as a parameter.

The ED I models have been investigated in [16, 18, 20] and four more ED I cubic nonlinearities (Models 9–12) are found in the present work. Thus, Models 2, 4, 7 (at $\sigma = 0$) and Models 9–12 can be termed as ED I models. In this context it is worth noting that while a universal two-point map is known for Model 2 (for arbitrary γ and δ), no JEF or any other analytical solutions are known so far (except at $\gamma = \delta = 0$). On the other hand, no universal two-point map is known for Models 9–12.

It is likely that the SW Model 5 [8] is an ED II model because it supports the well-known TI kink and the TI sine solutions derived in the present work (see equation (25)) but these solutions are derived from reduced equations, as shown in section 5.3. The reduced equations do not contain the integration constant. On the other hand, e.g., the five-periodic solution derived from the non-factorized SW model possesses the PN potential, as shown in section 7.1.

Before closing, we spell out some of the open problems.

(1) For the Models 9–12, can one obtain a unified general two-point map from which all solutions, including the JEF solutions can be derived? Note that, at the moment by starting from sn, cn, dn solutions, one can obtain three different maps from which only the respective solution can be obtained.
(2) For Model 2, while a general two-point map is known, to date no analytic solution is known (except at $\gamma = \delta = 0$) which is characterized by the two parameters C and ϕ_0. Can one find few analytic solutions in this model?
(3) Can one rigorously show that all short period, sine and staggered sine solutions for any ED I or ED II model, follow from the lower order equations?
(4) Can one rigorously prove that the Speight and Ward Model 5 is only an ED II and not an ED I model?
(5) There is a belief that all ED models (at least ED I models) must have some conserved quantity. Unfortunately, for Model 7 (at $\sigma = 0$) and for Models from 9 to 12 (which are all ED I models) no such conserved quantity is known at present. Can one find such a quantity or disprove the conjecture?
(6) While it has been demonstrated that no discrete model can simultaneously have conservation of P_1 and energy E, it is not known whether one can have a model where both P_2 and E can be simultaneously conserved. The obvious guess would be *no*. It would be nice to prove or disprove this conjecture.
(7) Can one find particular TI solutions for the discrete models that do not belong to the ED I class, i.e., finding the isolated TI solutions of the discrete models that are not

considered by many researchers as the ED models. In the present study we have given several examples of such solutions, for instance, the TI trigonometric solutions given in section 3.2. It is possible that isolated TI solutions may exist for many discrete models. Many of the isolated TI solutions result from factorized static problems and thus, finding various factorizations of the original static problem can be a method for their derivation.

Finally, the results obtained in this paper are easily extended to the case of the general nonlinear Schrödinger equation [21]. We hope to address these issues in a forthcoming publication.

Acknowledgments

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